



Brief paper

Stabilization by using artificial delays: An LMI approach[☆]

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ABSTRACT

Static output-feedback stabilization for the n th order vector differential equations by using artificial multiple delays is considered. Under assumption of the stabilizability of the system by a static feedback that depends on the output and its derivatives up to the order $n - 1$, a delayed static output-feedback is found that stabilizes the system. The conditions for the stability analysis of the resulting closed-loop system are given in terms of simple LMIs. It is shown that the LMIs are always feasible for appropriately chosen gains and small enough delays. Robust stability analysis in the presence of uncertain time-varying delays and stochastic perturbation of the system coefficients is provided. Numerical examples including chains of three and four integrators that are stabilized by static output-feedbacks with multiple delays illustrate the efficiency of the method.

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1. Introduction

It is well-known that some classes of systems (e.g. chains of integrators or oscillators, inverted pendulums) that cannot be stabilized by memoryless static output-feedbacks, can be stabilized by using static output-feedbacks with delays (French, Ilchmann, & Mueller, 2009; Karafyllis, 2008; Kharitonov, Niculescu, Moreno, & Michiels, 2005; Michiels & Niculescu, 2014; Niculescu & Michiels, 2004). The idea of feedback design in this case is usually based on the employing of a stabilizing feedback that depends on the output derivatives, and further approximation of the output derivatives (e.g. by finite differences). In the existing works it is proved that the resulting delayed static output-feedback stabilizes the system for small enough delays. However, efficient and simple conditions for the design and robustness analysis are missing.

The objective of the present paper is to fill this gap for systems that are governed by n th order vector differential equation and that can be stabilized by a static feedback that depends on the output and its derivatives up to the order $n - 1$. Some first results for $n = 2$ were obtained recently in Fridman and Shaikhet

(2016), where simple LMIs for robust stability analysis of the closed-loop delayed systems were derived. Comparatively to more general LMIs for stability analysis of time-delay systems provided e.g. in Gu, Kharitonov, and Chen (2003) and Seuret and Gouaisbaut (2013) (that may also be applicable to delay-induced stability), the conditions of Fridman and Shaikhet (2016) are essentially simpler leading in numerical examples to slightly more conservative results. Moreover, differently from Gu et al. (2003) and Seuret and Gouaisbaut (2013), the feasibility of LMIs was justified in Fridman and Shaikhet (2016) for small enough delays.

In the present paper, we suggest a new idea to represent the delayed outputs in the form of Taylor expansion with the *integral (Lagrange) form of the remainder*. This leads to novel controller design and robust stability analysis via a novel simple Lyapunov functional. For $n = 2$, the suggested Lyapunov functional is different from the one of Fridman and Shaikhet (2016) and leads to less restrictive conditions. However, as in Fridman and Shaikhet (2016), this method employs a Lyapunov functional depending on the state derivative that seems to be not applicable to the stochastic case.

For the stochastic case, we develop the model transformation-based analysis initiated in Borne, Kolmanovskii, and Shaikhet (2000) and Shaikhet (2013) and applied in Fridman and Shaikhet (2016). The feasibility of the resulting LMIs is justified for appropriately chosen gains and small enough delays. Extension to time-varying delays and stochastic perturbations is considered. Numerical examples including chains of three and four integrators illustrate the efficiency of the results.

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2. Problem formulation and preliminaries

Consider the n th order vector system

$$y^{(n)}(t) = \sum_{i=0}^{n-1} A_i y^{(i)}(t) + Bu(t), \tag{2.1}$$

where $y(t) = y^{(0)}(t) \in \mathbb{R}^k$ is the measurement, $y^{(i)}(t)$ is the i th derivative of $y(t)$, $u(t) \in \mathbb{R}^m$ is the control input, $A_i \in \mathbb{R}^{k \times k}$ and $B \in \mathbb{R}^{k \times m}$ are constant matrices. Assume that the open-loop system is unstable, and we are looking for a simple static output-feedback that will stabilize the system. It may happen that (2.1) is not stabilizable by $u(t) = K_0 y(t)$, but may be stabilizable by using artificial multiple delays (Karafyllis, 2008; Niculescu & Michiels, 2004).

Denote

$$\begin{aligned} x(t) &= \text{col}\{y^{(0)}(t), \dots, y^{(n-1)}(t)\} \\ &= \text{col}\{x_0(t), \dots, x_{n-1}(t)\}, \\ \bar{B} &= \text{col}\{0, \dots, 0, B\} \in \mathbb{R}^{nk \times m}, \\ A &= \begin{bmatrix} 0 & I_k & 0 & \dots & 0 \\ 0 & 0 & I_k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_k \\ A_0 & A_1 & A_2 & \dots & A_{n-1} \end{bmatrix} \in \mathbb{R}^{nk \times nk}. \end{aligned} \tag{2.2}$$

Then (2.1) can be presented as

$$\dot{x}(t) = Ax(t) + \bar{B}u(t). \tag{2.3}$$

Assume that the pair (A, \bar{B}) is stabilizable, i.e. there exists a matrix $\bar{K} = [\bar{K}_0 \dots \bar{K}_{n-1}] \in \mathbb{R}^{m \times nk}$ such that the matrix $D = A + \bar{B}\bar{K}$ is Hurwitz. The corresponding state-feedback has a form

$$u(t) = \sum_{j=0}^{n-1} \bar{K}_j x_j(t), \quad \bar{K}_j \in \mathbb{R}^{m \times k}.$$

Since the derivatives $y^{(j)}(t) = x_j(t)$, $j = 1, \dots, n-1$ are not available, we approximate them by using the delayed measurements $x_0(t - h_j)$ ($j = 1, \dots, n-1$), where

$$0 < h_1 < \dots < h_{n-1}. \tag{2.4}$$

Differently from Fridman and Shaikhet (2016), we employ in this paper the Taylor expansion with the integral form of the remainder:

$$x_0(t - h_i) = \sum_{j=0}^{n-1} \frac{1}{j!} (-h_i)^j x_j(t) + W_i(x_{nt}), \quad i = 1, \dots, n-1, \tag{2.5}$$

where

$$W_i(x_{nt}) = \frac{(-1)^n}{(n-1)!} \int_{t-h_i}^t (s-t+h_i)^{n-1} x_n(s) ds \tag{2.6}$$

and where $x_n(s) = \dot{x}_{n-1}(s)$. Note that $W_i(x_{nt}) = O(h_i^n)$. In Fridman and Shaikhet (2016) the delayed state was presented as $x_0(t - h_1) = x_0(t) - hx_1(t) + \delta(t)$ with $\delta(t) = O(h^2)$, whereas a particular form of the remainder δ was not exploited. In such a way it was not clear how to extend the results of Fridman and Shaikhet (2016) to $n > 2$.

Remark 2.1. For $n = 1$ representation (2.5) coincides with the basic relation

$$x_0(t - h_1) = x_0(t) - \int_{t-h_1}^t \dot{x}_0(s) ds$$

for delay-dependent stability conditions (see e.g. Fridman, 2014; Kolmanovskii & Myshkis, 1999). In this sense the Lyapunov-based analysis of Section 3 naturally extends simple delay-dependent conditions from the 1st order to the n th order systems.

Denoting $h_0 = 0$, we will find a delayed stabilizing static output-feedback

$$u(t) = \sum_{i=0}^{n-1} K_i x_0(t - h_i), \quad K_i \in \mathbb{R}^{r \times k}. \tag{2.7}$$

Substituting (2.7) into (2.3), we obtain the following closed-loop system with delays

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{n-1} \bar{B}K_i x_0(t - h_i). \tag{2.8}$$

From (2.5) we have

$$\sum_{i=0}^{n-1} K_i x_0(t - h_i) = \sum_{j=0}^{n-1} \bar{K}_j x_j(t) + \sum_{i=1}^{n-1} K_i W_i(x_{nt}), \tag{2.9}$$

where

$$\begin{aligned} \bar{K}_0 &= \sum_{i=0}^{n-1} K_i \quad \text{and} \\ \bar{K}_j &= \frac{(-1)^j}{j!} \sum_{i=1}^{n-1} h_i^j K_i, \quad j = 1, \dots, n-1. \end{aligned} \tag{2.10}$$

From (2.10) for $K = [K_0 \dots K_{n-1}]$ we obtain

$$\begin{aligned} \bar{K} &= KM, \\ M &= \begin{bmatrix} I_k & 0 & 0 & \dots & 0 \\ I_k & -h_1 I_k & \frac{h_1^2}{2} I_k & \dots & \frac{(-h_1)^{n-1}}{(n-1)!} I_k \\ \dots & \dots & \dots & \dots & \dots \\ I_k & -h_{n-1} I_k & \frac{h_{n-1}^2}{2} I_k & \dots & \frac{(-h_{n-1})^{n-1}}{(n-1)!} I_k \end{bmatrix}. \end{aligned} \tag{2.11}$$

Since all the delays are different, the Vandermonde-type matrix M is invertible. Moreover, the following holds:

Lemma 2.1. Let $h_i = ih$ ($i = 0, \dots, n-1$) for some $h > 0$ and M be given by (2.11). Then $M^{-1} = O(h^{-n+1})$, i.e. the absolute values of the entries of M^{-1} are bounded from above by Ch^{-n+1} with a positive constant $C = C(n)$.

Proof. The matrix M can be regarded as a matrix consisting of k equal Vandermonde-type blocks M_b of size n (each block is Vandermonde up to division of columns by corresponding factorials). In particular, the determinant of each block is given by

$$\begin{aligned} \det M_b &= C_1 \prod_{0 \leq i < j \leq n-1} (h_i - h_j) \\ &= C_1 (h_0 - h_1) \dots (h_0 - h_{n-1})(h_1 - h_2) \dots (h_1 - h_{n-1}) \\ &\quad \times \dots \times (h_{n-2} - h_{n-1}) = C_2 h^{n(n-1)/2} \end{aligned}$$

with C_1, C_2 being functions of n .

Similarly to M , the inverse M^{-1} consists of k inverse matrices $(M_b)^{-1}$. We can write $(M_b)^{-1} = \frac{1}{\det M_b} \text{Adj}(M_b)$, where the entries of $\text{Adj}(M_b)$ are $(n-1) \times (n-1)$ minors of M_b with some signs. Any $(n-1) \times (n-1)$ minor of M_b , regarded as the sum of products of elements taken one from each column, appears to be proportional to $h^{n(n-1)/2-s+1}$, where s is the number of the removed column of M_b . Thus, the minimal order of the $(n-1) \times (n-1)$ minors of M_b corresponds to the last removed column with $s = n$. Hence, the minimal order of an entry of M^{-1} is $(h^{\frac{n(n-1)}{2}-n+1})/h^{\frac{n(n-1)}{2}} = h^{-n+1}$. \square

Remark 2.2. As it is seen from the proof, the result of Lemma 2.1 remains true for more general choice of h_i that are different and such that $h_i = O(h)$ and $h_i - h_j = O(h)$ ($i \neq j$). The same is true concerning items (ii) of theorems below.

Substitution of (2.9) into (2.8) yields

$$\dot{x}(t) = D x(t) + \sum_{i=1}^{n-1} \bar{B} K_i W_i(x_{nt}), \quad (2.12)$$

$$D = A + \bar{B} \bar{K} = A + \bar{B} K M,$$

where M is given by (2.11).

As in Fridman and Shaikhet (2016), we suggest two different methods for stability analysis of the system (2.12). One method is based on the direct Lyapunov–Krasovskii analysis of (2.12), whereas the second one employs a neutral type model transformation. We will suppose that the matrix D is Hurwitz. Our stability analysis will employ the following Jensen’s inequality:

Lemma 2.2 (Jensen’s Inequality Solomon & Fridman, 2013). Denote

$$G = \int_a^b f(s) x(s) ds,$$

where $a \leq b, f : [a, b] \rightarrow [0, \infty), x(s) \in \mathbb{R}^n$ and the integration concerned is well defined. Then for any $n \times n$ matrix $R > 0$ the following inequality holds:

$$G' R G \leq \int_a^b f(\theta) d\theta \int_a^b f(s) x'(s) R x(s) ds. \quad (2.13)$$

3. Stability conditions: no model transformation

3.1. Constant delays

In this section we analyze stability of (2.8) by using its presentation (2.12).

Theorem 3.1. (i) Given $K_i \in \mathbb{R}^{m \times k}$ ($i = 0, \dots, n - 1$) and constant known delays $0 = h_0 < h_1 < \dots < h_{n-1}$ such that the matrix $D = A + \bar{B} K M$ with M defined by (2.11) and $K = [K_0 \dots K_{n-1}]$ is Hurwitz. Let there exist $0 < P \in \mathbb{R}^{n \times n}$ and $0 < R_i \in \mathbb{R}^{k \times k}, i = 1, \dots, n - 1$ that satisfy the LMI

$$\Psi_0 = \begin{bmatrix} \Phi_0 & X & \dots & X & D' [0_{k \times (n-1)k} \ I_k] \hat{R} \\ * & -(n!)^2 R_1 & \dots & 0 & B' \hat{R} \\ \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & -(n!)^2 R_{n-1} & B' \hat{R} \\ * & * & \dots & * & -\hat{R} \end{bmatrix} < 0, \quad (3.1)$$

where

$$\Phi_0 = P D' + D P \in \mathbb{R}^{n \times n}, \quad X = P \bar{B} \in \mathbb{R}^{n \times m}, \quad \hat{R} = \sum_{i=1}^{n-1} h_i^{2n} K_i' R_i K_i \in \mathbb{R}^{k \times k}. \quad (3.2)$$

Then system (2.8) is asymptotically stable.

(ii) Given $\bar{K} \in \mathbb{R}^{m \times nk}$ such that $D = A + \bar{B} \bar{K}$ is Hurwitz, and $h_i = ih$ ($i = 0, \dots, n - 1$) with some $h > 0$. Let $K = M^{-1} \bar{K}$, where M is given by (2.11). Then the LMI (3.1) is always feasible for small enough h , i.e. (2.8) is stabilizable by the delayed feedback (2.7) with small enough $h_i = ih$.

Proof. (i) Differentiating $V_1(x(t)) = x'(t) P x(t)$ ($P > 0$) along (2.12) we have

$$\frac{d}{dt} V_1(x_t) = 2x'(t) P \left(D x(t) + \sum_{i=1}^{n-1} \bar{B} K_i W_i(x_{nt}) \right). \quad (3.3)$$

To compensate in (3.3) the terms with $W_i(x_{nt})$, consider the functional

$$V_2(x_t) = \sum_{i=1}^{n-1} h_i^n \int_{t-h_i}^t (s-t+h_i)^n x_n'(s) K_i' R_i K_i x_n(s) ds,$$

where $R_i > 0$. Then

$$\frac{d}{dt} V_2(x_t) = \sum_{i=1}^{n-1} \left[h_i^{2n} x_n'(t) K_i' R_i K_i x_n(t) - n h_i^n \int_{t-h_i}^t (s-t+h_i)^{n-1} x_n'(s) K_i' R_i K_i x_n(s) ds \right]. \quad (3.4)$$

Via Jensen’s inequality (2.13) for $W_i(x_{nt})$ we find

$$W_i'(x_{nt}) K_i' R_i K_i W_i(x_{nt}) \leq \frac{h_i^n}{(n-1)! n!} \int_{t-h_i}^t (s-t+h_i)^{n-1} x_n'(s) K_i' R_i K_i x_n(s) ds$$

or

$$-n h_i^n \int_{t-h_i}^t (s-t+h_i)^{n-1} x_n'(s) K_i' R_i K_i x_n(s) ds \leq -(n!)^2 W_i'(x_{nt}) K_i' R_i K_i W_i(x_{nt}). \quad (3.5)$$

From (3.4), (3.5) by using notation (3.2) we obtain

$$\frac{d}{dt} V_2(x_t) \leq x_n'(t) \hat{R} x_n(t) - \sum_{i=1}^{n-1} (n!)^2 W_i'(x_{nt}) K_i' R_i K_i W_i(x_{nt}). \quad (3.6)$$

Consider the Lyapunov functional $V(x_t) = V_1(x(t)) + V_2(x_t)$. From (3.3) and (3.6) we obtain

$$\begin{aligned} \frac{d}{dt} V(x_t) &\leq x'(t) (P D' + D P) x(t) + x_n'(t) \hat{R} x_n(t) \\ &\quad + 2x'(t) X \sum_{i=1}^{n-1} K_i W_i(x_{nt}) \\ &\quad - \sum_{i=1}^{n-1} (n!)^2 W_i'(x_{nt}) \bar{R}_i W_i(x_{nt}) \\ &= \eta'(t) \mathcal{E} \eta(t) + x_n'(t) \hat{R} x_n(t). \end{aligned} \quad (3.7)$$

Here \mathcal{E} coincides with Ψ_0 , where the last column and row are deleted, and

$$\eta(t) = \text{col}\{x(t), K_1 W_1(x_{nt}), \dots, K_{n-1} W_{n-1}(x_{nt})\}.$$

Note that (2.12) yields

$$x_n(t) = [0_{k \times (n-1)k} \ I_k] D x(t) + \sum_{i=1}^{n-1} \bar{B} K_i W_i(x_{nt}).$$

Substituting this into the last term of (3.7) and applying Schur complement we conclude that $\frac{d}{dt} V(x_t) \leq -c|x(t)|^2$ for some $c > 0$ if $\Psi_0 < 0$. So, (2.12) (and, thus, (2.8)) is asymptotically stable. The proof of (i) is completed.

(ii) By Lemma 2.1 we have $M^{-1} = O(h^{-n+1})$ implying $K = O(h^{-n+1})$. Let $P > 0$ be a solution of the Lyapunov equation

$$D'P + PD = -I_{nk}.$$

Choose $R_i = h^{-1}I_m$. Then $\hat{R} = O(h^{-1})$. Applying Schur complements to $\Psi_0 < 0$, we arrive at

$$D'P + PD + O(h) < 0,$$

that holds for small enough h . The proof of (ii) is completed. \square

Remark 3.1. Theorem 3.1 improves the corresponding result in Fridman and Shaikhet (2016) due to novel Lyapunov functional introduced in the proof. For $n = 2$ the term V_2 used in the proof of Theorem 3.1 has a form

$$V_2(x_t) = h_1^2 \int_{t-h_1}^t (s-t+h_1)^2 x_2'(s) K_1' R_1 K_1 x_2(s) ds,$$

whereas in Fridman and Shaikhet (2016) the following term was employed

$$\tilde{V}_2(x_t) = h_1 \int_{t-h_1}^t (s-t+h_1) x_2'(s) K_1' R_1 K_1 x_2(s) ds.$$

The term \tilde{V}_2 leads to LMI (3.1), where the (2,2)-diagonal term is replaced by $-0.25\pi^2 R_1$ which is greater than $-4R_1$ in (3.1), i.e. the LMI of Fridman and Shaikhet (2016) is more restrictive.

Remark 3.2. The representation (2.5) is used with the same n for all h_i . The latter allows to derive feasible LMIs. The feasibility claims (for given \bar{K} and not for given K) and their proofs in the present paper are different from the ones in Fridman and Shaikhet (2016) and are based on Lemma 2.1 and on the feasibility of the Lyapunov equation for the closed-loop system under the full state-feedback.

Remark 3.3. The results of Theorem 3.1 can be extended to performance analysis, e.g. to L_2 -gain analysis of the perturbed system with a disturbance $v \in L_2[0, \infty)$ and with the noisy measurements $y = x_0 + Ev$, where E has appropriate dimensions. In this case the delayed feedback (2.7) will contain the delayed disturbance:

$$u(t) = \sum_{i=0}^{n-1} K_i [x_0(t-h_i) + Ev(t-h_i)].$$

Let z be a controlled output. To derive LMI conditions for the L_2 -gain analysis of the resulting closed-loop system, one can find conditions for

$$\dot{V} + |z|^2 - \sum_{i=0}^{n-1} \gamma_i^2 |v(t-h_i)|^2 \leq 0, \quad t \geq 0$$

with $\sum_{i=0}^{n-1} \gamma_i^2 \leq \gamma^2$, where $v(t) = 0$ for $t < 0$. This condition guarantees that the L_2 -gain of the closed-loop system is less or equal to γ since

$$\begin{aligned} \sum_{i=0}^{n-1} \gamma_i^2 \int_0^\infty |v(t-h_i)|^2 dt &= \sum_{i=0}^{n-1} \gamma_i^2 \int_0^\infty |v(t)|^2 dt \\ &\leq \gamma^2 \int_0^\infty |v(t)|^2 dt. \end{aligned}$$

3.2. Extension to time-varying delays

For practical application of a delayed static output feedback, consider its sampled-data implementation. Let $0 = t_0 < t_1 < \dots < t_k < \dots$, $k = 0, 1, \dots$ be sampling instants $t_k = kT$, where

$T > 0$ is the sampling period, and let $\eta_k \in [0, \eta_M]$ be variable and unknown input delay with a known upper bound η_M . Then a sampled-data delayed controller can be presented as

$$u(t) = \sum_{i=0}^{n-1} K_i x_0(t_{k-ip} - \eta_k), \quad t \in [t_k, t_{k+1}] \tag{3.8}$$

where p is some natural number (the sampling delay). Using the time-delay approach to sampled-data control (Fridman, 2014; Fridman, Seuret, & Richard, 2004; Liu & Fridman, 2012), the latter controller may be further presented as the delayed one (2.7) with time-varying delays:

$$h_i(t) = t - t_{k-ip} + \eta_k, \quad i = 0, \dots, n-1, \quad t \in [t_k, t_{k+1}].$$

Clearly $h_i(t) \in [ipT, ipT + T + \eta_M]$.

In the case of additional (unknown and bounded) measurement delay, the closed-loop sampled-data system can be also presented as a system with time-varying delays, where delays belong to some intervals. Motivated by sampled-data implementation, in this section we provide stability analysis of the closed-loop system (2.8) with time-varying and unknown delays $h_i = h_i(t)$, such that $h_i(t) \in [h_{im}, h_{iM}]$, $h_{iM} \geq h_{im} \geq 0$, $i = 0, \dots, n-1$, $h_{0m} = 0$. We have

$$x_0(t - h_i(t)) = x_0(t - h_{im}) + \delta_i(t),$$

where

$$\delta_i(t) = \int_{t-h_{im}}^{t-h_i(t)} x_1(s) ds, \quad i = 0, \dots, n-1. \tag{3.9}$$

We employ (2.5). Then from (2.9), (2.10) with $h_i = h_{im}$ we obtain

$$\sum_{i=0}^{n-1} K_i x_0(t - h_i(t)) = \sum_{j=0}^{n-1} \bar{K}_j x_j(t) + \sum_{i=1}^{n-1} K_i W_i(x_{nt}) + \sum_{i=0}^{n-1} K_i \delta_i(t), \tag{3.10}$$

where W_i and \bar{K}_j are given by (2.6) and (2.10) respectively with h_i replaced by h_{im} . So, instead of (2.12) for constant delays, we obtain the following presentation of (2.8) with time-varying delays:

$$\dot{x}(t) = Dx(t) + \sum_{i=1}^{n-1} \bar{B} K_i W_i(x_{nt}) + \sum_{i=0}^{n-1} \bar{B} K_i \delta_i(t). \tag{3.11}$$

Here $D = A + \bar{B} K M|_{h_i=h_{im}}$ with M given by (2.11). For the function $V_1(x(t)) = x'(t) P x(t)$ similarly to (3.3) we have

$$\frac{d}{dt} V_1(x_t) = 2x'(t) P \left(Dx(t) + \sum_{i=1}^{n-1} \bar{B} K_i W_i(x_{nt}) + \sum_{i=0}^{n-1} \bar{B} K_i \delta_i(t) \right). \tag{3.12}$$

To compensate in (3.12) the terms with $W_i(x_{nt})$ and $\delta_i(t)$, consider the simplest functional for this case

$$\begin{aligned} V_2(x_t) &= \sum_{i=1}^{n-1} h_{im}^n \int_{t-h_{im}}^t (s-t+h_{im})^n x_n'(s) K_i' R_i K_i x_n(s) ds \\ &+ \sum_{i=0}^{n-1} (h_{iM} - h_{im}) \int_{t-h_{iM}}^{t-h_{im}} (s-t+h_{im}) x_1'(s) \bar{U}_i x_1(s) ds \\ &+ \sum_{i=1}^{n-1} (h_{iM} - h_{im})^2 \int_{t-h_{im}}^t x_1'(s) \bar{U}_i x_1(s) ds, \\ \bar{U}_i &= K_i' U_i K_i, \quad R_i, U_i > 0. \end{aligned}$$

Note that the results for time-varying delays via the simplest Lyapunov functionals (with just \bar{U}_i "double integral terms") may be

restrictive, but they may be improved by using advanced methods for time-varying delays that lead to larger LMIs (see e.g. Fridman, 2014; Park, Ko, & Jeong, 2011). Differentiating $V_2(x_t)$ we have

$$\begin{aligned} \frac{d}{dt}V_2(x_t) &= \sum_{i=1}^{n-1} h_{im}^n \left(h_{im}^n x_n'(t) K_i' R_i K_i x_n(t) \right. \\ &\quad - n \int_{t-h_{im}}^t (s-t+h_{im})^{n-1} x_n'(s) K_i' R_i K_i x_n(s) ds \\ &\quad - \sum_{i=0}^{n-1} (h_{iM} - h_{im}) \int_{t-h_{iM}}^{t-h_{im}} x_1'(s) \bar{U}_i x_1(s) ds \\ &\quad \left. + \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 x_1'(t) \bar{U}_i x_1(t) \right). \end{aligned}$$

Using Jensen’s inequality, we obtain

$$-(h_{iM} - h_{im}) \int_{t-h_{iM}}^{t-h_{im}} x_2'(s) \bar{U}_i x_2(s) ds \leq -\delta_i'(t) \bar{U}_i \delta_i(t), \quad (3.13)$$

and (3.5) with h_i replaced by h_{im} . Then for $\dot{V}_2(x_t)$ instead of (3.6) we have

$$\begin{aligned} \frac{d}{dt}V_2(x_t) &\leq x_n'(t) \hat{R} x_n(t) + \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 x_1'(t) \bar{U}_i x_1(t) \\ &\quad - (n!)^2 \sum_{i=0}^{n-1} W_i'(x_{nt}) K_i' R_i K_i W_i(x_{nt}) - \sum_{i=0}^{n-1} \delta_i'(t) \bar{U}_i \delta_i(t), \quad (3.14) \end{aligned}$$

where $\hat{R} = \sum_{i=1}^{n-1} h_{im}^{2n} K_i' R_i K_i$. From (3.12), (3.14) for the functional $V(x_t) = V_1(x_t) + V_2(x_t)$ we find

$$\begin{aligned} \frac{d}{dt}V(x_t) &\leq 2x'(t) P D x(t) + x_n'(t) \hat{R} x_n(t) \\ &\quad + 2x'(t) P \bar{B} \left(\sum_{i=1}^{n-1} K_i W_i(x_{nt}) + \sum_{i=0}^{n-1} K_i \delta_i(t) \right) \\ &\quad + \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 x_1'(t) \bar{U}_i x_1(t) \\ &\quad - (n!)^2 \sum_{i=1}^{n-1} W_i'(x_{nt}) K_i' R_i K_i W_i(x_{nt}) \\ &\quad - \sum_{i=0}^{n-1} \delta_i'(t) \bar{U}_i \delta_i(t). \quad (3.15) \end{aligned}$$

Comparing (3.15) with (3.7) we arrive at the following generalization of Theorem 3.1:

Theorem 3.2. (i) Given $K_i \in \mathbb{R}^{m \times k}$, $h_{im} \geq h_{im} \geq 0$ ($i = 0, \dots, n-1$) and $h_{0m} = 0$, assume that the matrix $D = A + \bar{B} K M|_{h_i=h_{im}}$ with M given by (2.11) and $K = [K_0 \dots K_{n-1}]$ is Hurwitz. If there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$, $R_i \in \mathbb{R}^{m \times m}$ ($i = 1, \dots, n-1$), and $U_j \in \mathbb{R}^{m \times m}$ ($j = 0, \dots, n-1$) that satisfy the LMI which is given in Box 1 where

$$\begin{aligned} \Phi_1 &= P D + D' P \\ &\quad + \text{diag} \left\{ 0_{k \times k}, \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 K_i' U_i K_i, 0_{k \times k}, \dots, 0_{k \times k} \right\}, \\ X &= P \bar{B}, \quad \hat{R} = \sum_{i=1}^{n-1} h_{im}^{2n} K_i' R_i K_i, \end{aligned}$$

then system (2.8) with time-varying delays is asymptotically stable.

(ii) Given $\bar{K} \in \mathbb{R}^{m \times nk}$ such that $D = A + \bar{B} \bar{K}$ is Hurwitz, and $h_{im} = ih$ ($i = 0, \dots, n-1$) with some $h > 0$, let $K = M^{-1} \bar{K}$, where M is given by (2.11) with $h_i = h_{im}$. Assume that $h_{iM} - h_{im} = O(h^n)$. Then the LMI $\Psi_1 < 0$ is always feasible for small enough h .

Proof. Item (i) was already proved.

(ii) By Lemma 2.1 $M^{-1} = O(h^{-n+1})$ implying $K = O(h^{-n+1})$. Let $P > 0$ be a solution of the Lyapunov equation $D'P + PD = -I_{nk}$. Choose $R_i = U_i = h^{-1} I_m$. Then $\hat{R} = O(h^{-1})$. Applying Schur complements to $\Psi_1 < 0$, we arrive at $D'P + PD + O(h) < 0$ that holds for small enough h . \square

4. Model transformation-based approach

In this section the system (2.8) will be presented in the form of neutral type system. This approach will be applied also to systems under stochastic perturbations.

4.1. Constant delays

Let us show that the integral $W_i(x_{nt})$ defined in (2.6) can be represented as derivative in time of the functional

$$\begin{aligned} G_i(x_{n-1,t}) &= \frac{(-1)^n}{(n-1)!} \int_{t-h_i}^t (s-t+h_i)^{n-1} x_{n-1}(s) ds, \\ &\quad i = 1, \dots, n-1. \quad (4.1) \end{aligned}$$

Indeed, differentiating $G_i(x_{n-1,t})$ and integrating by parts, via $\dot{x}_{n-1} = x_n$ and (2.6) we have

$$\begin{aligned} \frac{d}{dt}G_i(x_{n-1,t}) &= \frac{(-1)^n}{(n-1)!} \left[h_i^{n-1} x_{n-1}(t) \right. \\ &\quad \left. - (n-1) \int_{t-h_i}^t (s-t+h_i)^{n-2} x_{n-1}(s) ds \right] \\ &= \frac{(-1)^n}{(n-1)!} \left[h_i^{n-1} x_{n-1}(t) - \int_{t-h_i}^t x_{n-1}(s) d(s-t+h_i)^{n-1} \right] \\ &= \frac{(-1)^n}{(n-1)!} \left[h_i^{n-1} x_{n-1}(t) - \left(h_i^{n-1} x_{n-1}(t) \right. \right. \\ &\quad \left. \left. - \int_{t-h_i}^t (s-t+h_i)^{n-1} dx_{n-1}(s) \right) \right] = W_i(x_{nt}). \quad (4.2) \end{aligned}$$

Via (4.2) instead of (2.5) we obtain the representation

$$x_0(t-h_i) = \sum_{j=0}^{n-1} \frac{(-h_i)^j}{j!} x_j(t) + \frac{d}{dt}G_i(x_{n-1,t}) \quad (4.3)$$

that was used in Shaikhet (2013) for the stability analysis of systems of the type of (2.8).

Thus, the system (2.12) can be represented in the form

$$\dot{z}(t) = D x(t), \quad (4.4)$$

where

$$z(t) = x(t) - \sum_{i=1}^{n-1} \bar{B} K_i G_i(x_{n-1,t}), \quad (4.5)$$

and D is defined in (2.12).

The stability of the transformed neutral type systems, guarantees the stability of the original one (2.8). In order to use the Lyapunov–Krasovskii theorem for the stability of the neutral type systems (see e.g. Theorem 8.1 on p. 293 of Hale & Verduyn Lunel, 1993), we first derive conditions for the exponential stability of the

$$\Psi_1 = \begin{bmatrix} \Phi_1 & X & \cdots & X & X & \cdots & X & D'[0 \ I_k]'\hat{R} \\ * & -(n!)^2 R_1 & \cdots & 0 & 0 & \cdots & 0 & B'\hat{R} \\ \cdots & \cdots \\ * & * & \cdots & 0 & 0 & \cdots & 0 & B'\hat{R} \\ * & * & \cdots & -(n!)^2 R_{n-1} & 0 & \cdots & 0 & B'\hat{R} \\ * & * & \cdots & * & -U_0 & \cdots & 0 & B'\hat{R} \\ \cdots & \cdots \\ * & * & \cdots & * & * & \cdots & -U_{n-1} & B'\hat{R} \\ * & * & \cdots & * & * & \cdots & * & -\hat{R} \end{bmatrix} < 0,$$

Box I.

corresponding integral equation $z(t) = 0$. It is seen from (4.5) that the stability of $z(t) = 0$ is equivalent to the stability of

$$x_{n-1}(t) = \sum_{i=1}^{n-1} BK_i G_i(x_{n-1,t}). \tag{4.6}$$

Similarly to Fridman and Shaikhet (2016) the following statement can be proved:

Lemma 4.1. *Let there exist positive definite matrices $S_i \in \mathbb{R}^{k \times k}$ such that the following LMI holds*

$$\hat{K}'B' \left(\sum_{i=1}^{n-1} h_i^{2n} S_i \right) B\hat{K} - \bar{S} < 0, \tag{4.7}$$

where $\hat{K} = [K_1 \dots K_{n-1}] \in \mathbb{R}^{k \times (n-1)k}$ and

$$\bar{S} = \text{diag}\{(n!)^2 S_1, \dots, (n!)^2 S_{n-1}\} \in \mathbb{R}^{(n-1)k \times (n-1)k}.$$

Then the integral equation (4.6) is exponentially stable.

Remark 4.1. It can be seen that a simple sufficient condition for the feasibility of (4.7) (i.e. for the exponential stability of (4.6)) is given by $\sum_{i=1}^{n-1} h_i^n |BK_i| < n!$, that was used for stability analysis of neutral type systems in Kolmanovskii and Myshkis (1999) and Shaikhet (2013).

Theorem 4.1. (i) *Given $K_i \in \mathbb{R}^{m \times k}$ ($i = 0, \dots, n-1$) and constant known delays $0 = h_0 < h_1 < \dots < h_{n-1}$ such that the matrix $D = A + BKM$ with M defined by (2.11) and $K = [K_0 \dots K_{n-1}]$ is Hurwitz. Let there exist positive definite matrices $P \in \mathbb{R}^{nk \times nk}$ and $S_i \in \mathbb{R}^{k \times k}$, $R_i \in \mathbb{R}^{m \times m}$ ($i = 1, \dots, n-1$) that satisfy LMIs (4.7) and*

$$\Psi_2 = \begin{bmatrix} \Phi_2 & Z & \cdots & Z \\ * & -(n!)^2 R_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & 0 \\ * & * & \cdots & -(n!)^2 R_{n-1} \end{bmatrix} < 0, \tag{4.8}$$

where $Z = D'P\bar{B} \in \mathbb{R}^{nk \times m}$ and

$$\Phi_2 = D'P + PD + \text{diag}\left\{0_{k \times k}, \dots, 0_{k \times k}, \sum_{i=1}^{n-1} h_i^{2n} K_i' R_i K_i\right\}.$$

Then the system (2.8) is asymptotically stable.

(ii) *Given $\bar{K} \in \mathbb{R}^{m \times nk}$ such that $D = A + B\bar{K}$ is Hurwitz, and $h_i = ih$ ($i = 0, \dots, n-1$) with some $h > 0$, let $K = M^{-1}\bar{K}$, where M is given by (2.11). Then the LMI $\Psi_2 < 0$ is always feasible for small enough h .*

Proof. (i) By Lemma 4.1, LMI (4.7) guarantees the exponential stability of the integral equation (4.6). Differentiating

$$V_1(x_t) = z'(t)Pz(t), \quad P > 0, P \in \mathbb{R}^{nk \times nk} \tag{4.9}$$

along (4.4) and (4.5) we have

$$\begin{aligned} \frac{d}{dt} V_1(x_t) &= 2z'(t)P\dot{z}(t) \\ &= 2\left(x(t) - \sum_{i=1}^{n-1} \bar{B}K_i G_i(x_{n-1,t})\right)' PDx(t) \\ &= 2x'(t)PDx(t) - 2 \sum_{i=1}^{n-1} G_i'(x_{n-1,t})K_i' Z' x(t). \end{aligned} \tag{4.10}$$

In order to compensate in (4.10) the terms $G_i(x_{n-1,t})$, consider the additional functional

$$\begin{aligned} V_2(x_t) &= \sum_{i=1}^{n-1} h_i^n \int_{t-h_i}^t (s-t+h_i)^n x_{n-1}'(s) \hat{R}_i x_{n-1}(s) ds, \\ \hat{R}_i &= K_i' R_i K_i, \quad R_i > 0. \end{aligned}$$

Differentiating $V_2(x_t)$ and applying Jensen's inequality, similarly to (3.5) and (3.6), we obtain

$$\begin{aligned} \frac{d}{dt} V_2(x_t) &\leq \sum_{i=1}^{n-1} \left(h_i^{2n} x_{n-1}'(t) \bar{R}_i x_{n-1}(t) \right. \\ &\quad \left. - (n!)^2 G_i'(x_{n-1,t}) \bar{R}_i G_i(x_{n-1,t}) \right). \end{aligned} \tag{4.11}$$

From (4.10), (4.11) for the Lyapunov functional $V(x_t) = V_1(x_t) + V_2(x_t)$ we arrive at

$$\begin{aligned} \frac{d}{dt} V(x_t) &\leq \eta_2'(t) \Psi_2 \eta_2(t), \\ \eta_2(t) &= \text{col}\{x(t), -K_1 G_1(x_{n-1,t}), \dots, -K_{n-1} G_{n-1}(x_{n-1,t})\}. \end{aligned} \tag{4.12}$$

So, under (4.8), the system (4.4), (4.5) (and, thus, (2.8)) is asymptotically stable (Hale & Verduyn Lunel, 1993).

The proof of (ii) is similar to (ii) of Theorem 3.1. \square

4.2. Time-varying delays and stochastic perturbations

In this section we consider the system (2.8) with time-varying and unknown delays $h_i = h_i(t)$, such that $h_i(t) \in [h_{im}, h_{iM}]$, $h_{iM} \geq h_{im} \geq 0$, $i = 0, \dots, n-1$. Let $\delta_i(t)$ be given by (3.9), and $G_i(x_{n-1,t})$ be given by (4.1) with h_{im} instead of h_i . Similarly to (3.10) we obtain

$$\begin{aligned} \sum_{i=0}^{n-1} K_i x_0(t-h_i(t)) &= \sum_{i=1}^{n-1} K_i \frac{d}{dt} G_i(x_{n-1,t}) + \sum_{i=0}^{n-1} K_i \delta_i(t) \\ &\quad + \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \sum_{i=0}^{n-1} h_{im}^j K_i x_j(t). \end{aligned} \tag{4.13}$$

Then (2.8) takes the form

$$\dot{z}(t) = D x(t) + \sum_{i=0}^{n-1} \bar{B} K_i \delta_i(t), \tag{4.14}$$

where $D = A + \bar{B} K M|_{h_i=h_{im}}$ and $z(t)$ is defined in (4.5).

LMI (4.7) guarantees the exponential stability of integral equation (4.6). Differentiating (4.9) along (4.14), (4.5) we have

$$\begin{aligned} \frac{d}{dt} V_1(x_t) &= 2 \left(x(t) - \sum_{i=1}^{n-1} \bar{B} K_i G_i(x_{n-1,t}) \right)' P \\ &\quad \times \left(D x(t) + \sum_{i=0}^{n-1} \bar{B} K_i \delta_i(t) \right) \\ &= 2 x'(t) P D x(t) - 2 \sum_{i=1}^{n-1} G_i'(x_{n-1,t}) K_i' Z' x(t) \\ &\quad + 2 x'(t) X \sum_{i=0}^{n-1} K_i \delta_i(t) \\ &\quad - 2 \sum_{j=1}^{n-1} G_j'(x_{n-1,t}) K_j' Y \sum_{i=0}^{n-1} K_i \delta_i(t). \end{aligned} \tag{4.15}$$

In order to compensate in (4.15) the terms $G_j(x_{n-1,t})$ and $\delta_i(t)$ consider the additional functional

$$\begin{aligned} V_2(x_t) &= \sum_{i=1}^{n-1} h_{im}^n \int_{t-h_{im}}^t (s-t+h_{im})^n \\ &\quad \times x'_{n-1}(s) \bar{R}_i x_{n-1}(s) ds \\ &\quad + \sum_{i=0}^{n-1} (h_{iM} - h_{im}) \int_{t-h_{iM}}^{t-h_{im}} (s-t+h_{im}) \\ &\quad \times x'_1(s) \bar{U}_i x_1(s) ds \\ &\quad + \sum_{i=1}^{n-1} (h_{iM} - h_{im})^2 \int_{t-h_{im}}^t x'_1(s) \bar{U}_i x_1(s) ds, \\ \bar{R}_i &= K_i' R_i K_i, \quad \bar{U}_i = K_i' U_i K_i, \quad R_i, U_i > 0. \end{aligned} \tag{4.16}$$

Differentiating $V_2(x_t)$ and applying Jensen's inequality (cf. (3.5) and (3.13)), similarly to (3.14), we obtain

$$\begin{aligned} \frac{d}{dt} V_2(x_t) &\leq \sum_{i=1}^{n-1} h_{im}^{2n} x'_{n-1}(t) \bar{R}_i x_{n-1}(t) \\ &\quad + \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 x'_1(t) \bar{U}_i x_1(t) \\ &\quad - (n!)^2 \sum_{i=1}^{n-1} G_i'(x_{n-1,t}) \bar{R}_i G_i(x_{n-1,t}) \\ &\quad - \sum_{i=0}^{n-1} \delta_i'(t) \bar{U}_i \delta_i(t). \end{aligned} \tag{4.17}$$

From (4.15), (4.17) for the Lyapunov functional $V(x_t) = V_1(x_t) + V_2(x_t)$ we find

$$\begin{aligned} \frac{d}{dt} V(x_t) &\leq 2 x'(t) P D x(t) \\ &\quad - 2 \sum_{i=1}^{n-1} G_i'(x_{n-1,t}) K_i' Z' x(t) + 2 x'(t) X \sum_{i=0}^{n-1} K_i \delta_i(t) \\ &\quad - 2 \sum_{j=1}^{n-1} G_j'(x_{n-1,t}) K_j' Y \sum_{i=0}^{n-1} K_i \delta_i(t) \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=1}^{n-1} h_{im}^{2n} x'_{n-1}(t) \bar{R}_i x_{n-1}(t) \\ &+ \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 x'_1(t) \bar{U}_i x_1(t) - \sum_{i=0}^{n-1} \delta_i'(t) \bar{U}_i \delta_i(t) \\ &- (n!)^2 \sum_{i=1}^{n-1} G_i'(x_{n-1,t}) \bar{R}_i G_i(x_{n-1,t}) \leq \eta'_3(t) \Psi_3 \eta_3(t), \end{aligned} \tag{4.18}$$

where

$$\eta_3(t) = \text{col}\{x(t), -K_1 G_1(x_{n-1,t}), \dots, -K_{n-1} G_{n-1}(x_{n-1,t}), K_0 \delta_0, \dots, K_{n-1} \delta_{n-1}\}$$

and

$$\Psi_3 = \begin{bmatrix} \Phi_3 & Z & \dots & Z & X & \dots & X \\ * & -(n!)^2 R_1 & \dots & 0 & Y & \dots & Y \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & 0 & Y & \dots & Y \\ * & * & \dots & -(n!)^2 R_{n-1} & Y & \dots & Y \\ * & * & \dots & * & -U_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & * & * & \dots & -U_{n-1} \end{bmatrix},$$

$$\begin{aligned} \Phi_3 &= P D + D' P + \text{diag}\left\{0_{k \times k}, \sum_{i=0}^{n-1} (h_{iM} - h_{im})^2 K_i' U_i K_i, \right. \\ &\quad \left. 0_{k \times k}, \dots, 0_{k \times k}, \sum_{i=1}^{n-1} h_{im}^{2n} K_i' R_i K_i\right\}, \\ X &= P \bar{B}, \quad Y = \bar{B}' X, \quad Z = D' X. \end{aligned} \tag{4.19}$$

Suppose now that the coefficients A_0, \dots, A_{n-1} in (2.1) are under the influence of stochastic perturbations of the type of white noise, i.e., $A_i \rightarrow A_i + \sigma_i \dot{w}_i(t)$, $i = 0, 1, \dots, n-1$, where $\sigma_i \in \mathbb{R}^{k \times k \times q_i}$ are constant matrices, $w_i(t) \in \mathbb{R}^{q_i}$ are mutually independent Wiener processes with independent components (Gikhman & Skorokhod, 1972; Shaikhet, 2013). In this case instead of (4.14) we obtain the system of Ito's stochastic differential equations

$$dz(t) = \left(D x(t) + \sum_{i=0}^{n-1} \bar{B} K_i \delta_i(t) \right) dt + \mathcal{C}(x(t)) dw(t), \tag{4.20}$$

where $z(t)$ is defined in (4.5) and

$$\begin{aligned} \mathcal{C}(x(t)) &= \text{col}\{0, \sigma(x(t))\} \in \mathbb{R}^{nk \times q}, \quad q = \sum_{i=0}^{n-1} q_i, \\ \sigma(x(t)) &= ((\sigma_0 x_0(t)) \dots (\sigma_{n-1} x_{n-1}(t))) \in \mathbb{R}^{k \times q}, \\ w(t) &= \text{col}\{w_0(t), \dots, w_{n-1}(t)\} \in \mathbb{R}^q. \end{aligned} \tag{4.21}$$

Denote by L the generator of the stochastic differential equation (4.20) (see Gikhman & Skorokhod, 1972; Shaikhet, 2013). Then for Lyapunov functional $V(x_t) = z'(t) P z(t) + V_2(x_t)$, where V_2 is given by (4.16), similarly to (4.18) we obtain

$$\begin{aligned} L V(x_t) &\leq \eta'_3(t) \Psi_3 \eta_3(t) + \text{Tr}[\mathcal{C}'(x(t)) P \mathcal{C}(x(t))] \\ &= \eta'_3(t) \Psi_3 \eta_3(t) + \sum_{i=0}^{n-1} x'_i(t) T_i x_i(t), \end{aligned}$$

where

$$T_i = \text{Tr}[\sigma_i' [0 \dots 0 I_k] P [0 \dots 0 I_k] \sigma_i] \in \mathbb{R}^{k \times k}. \tag{4.22}$$

Thus, we arrive at the following result:

Theorem 4.2. (i) Given $K_i \in \mathbb{R}^{m \times k}$, $h_{iM} \geq h_{im} \geq 0$ ($i = 0, \dots, n-1$) and $h_{0M} = 0$, assume that the matrix $D = A + \bar{B} K M|_{h_i=h_{im}}$

with M given by (2.11) and $K = [K_0 \dots K_{n-1}]$ is Hurwitz. If there exist positive definite matrices $P \in \mathbb{R}^{nk \times nk}$, $S_i \in \mathbb{R}^{k \times k}$ and $R_i, U_j \in \mathbb{R}^{m \times m}$ ($i = 1, \dots, n-1, j = 0, \dots, n-1$) that satisfy LMIs (4.7) and

$$\Psi_3 + \text{diag}\{T_0, \dots, T_{n-1}, 0_{2nm \times 2nm}\} < 0, \tag{4.23}$$

where Ψ_3 is defined by (4.19) with $D = D|_{h_i=h_{im}}$ and T_i are defined by (4.22), then the system (4.20), (4.21), (4.5) is asymptotically mean square stable.

- (ii) Given $\bar{K} \in \mathbb{R}^{m \times nk}$ such that $D = A + \bar{B}\bar{K}$ is Hurwitz, $h_{im} = ih$, $h_{iM} - h_{im} = O(h^n)$ and $\sigma_i = O(\sqrt{h})$ ($i = 0, \dots, n-1$) with some $h > 0$, let $K = M^{-1}\bar{K}$, where M is given by (2.11) with $h_i = h_{im}$. Then the LMI (4.23) is always feasible for small enough h .

5. Examples

5.1. Scalar system of the third order

In this section we consider (2.1) with $n = 3, k = m = 1$ and $B = 1$. The matrix D here has a form

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -Q_0 & -Q_1 & -Q_2 \end{bmatrix} \tag{5.1}$$

with

$$\begin{aligned} Q_0 &= -A_0 - (K_0 + K_1 + K_2), \\ Q_1 &= -A_1 + h_1K_1 + h_2K_2, \\ Q_2 &= -A_2 - \frac{1}{2}(h_1^2K_1 + h_2^2K_2). \end{aligned} \tag{5.2}$$

The matrix (5.1) is Hurwitz if and only if (see e.g. Shaikhet, 2013)

$$Q_i > 0, \quad i = 0, 1, 2, \quad Q_2Q_1 > Q_0. \tag{5.3}$$

In the examples below (for $n = 3$), stabilizing gains K_i and h_1, h_2 of the delayed feedback (2.7) are obtained via conditions (5.3) that guarantee D to be a Hurwitz matrix.

Example 5.1. Consider the chain of three integrators, where

$$A_0 = A_1 = A_2 = 0. \tag{5.4}$$

Choose

$$\begin{aligned} h_0 &= 0, & h_1 &= h, & h_2 &= 2h, \\ K_0 &= -1.1002, & K_1 &= 2.1, & K_2 &= -1 \end{aligned} \tag{5.5}$$

that lead to Hurwitz matrix D . By Theorem 3.1, the resulting closed-loop system (2.8) is asymptotically stable for $h \in [0.127, 0.843]$. Conditions of Theorem 4.1 lead to a slightly smaller stability interval $h \in [0.130, 0.840]$. Simulations of solutions of the system with the initial function $y(s) = -0.8 \cos(s)$ ($s \in [-2h, 0]$) are shown in Fig. 1 for different delays: $h = 0.15, h = 0.40, h = 0.85$ and $h = 1.209$. One can see that for $h = 1.209$ the system is unstable, meaning that the LMI conditions are efficient (not too conservative).

Consider now the sampled-data implementation (3.8) of the delayed controller with the sampling period T , sampling delay p and unknown input delay bounded by η_M . We apply Theorems 3.2 and 4.2 with

$$h_{im} = ipT, \quad h_{iM} = ipT + T + \eta_M, \quad i = 0, 1, 2. \tag{5.6}$$

Theorem 4.2 is applied either to stochastic case in the presence of noise $A_1 = \sigma_1 \dot{w}(t)$ with $\sigma_1 = 0.3$ or to deterministic case, where $\sigma_1 = 0$, whereas Theorem 3.2 is applicable only to deterministic case. As it is seen from Table 1, Theorem 4.2 here leads to more efficient results (larger T and smaller p for small η_M , or larger η_M for the same T and p).

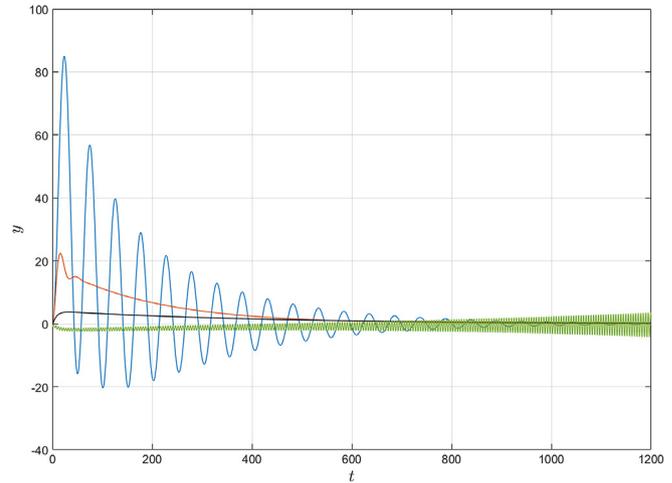


Fig. 1. Solutions of (2.8), (5.4), (5.5) for different delays: $h = 0.15$ (blue), $h = 0.40$ (red), $h = 0.85$ (black), $h = 1.209$ (green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Example 5.1: stabilizing T and p for digital implementation.

Theorems	T	p	η_M	σ_1
Theorem 3.2	0.0132	50	0.00041	0
Theorem 4.2	0.0195	43	0.00027	0
Theorem 4.2	0.0195	43	0.000002	0.03
Theorem 3.2	0.0028	293	0.0025	0
Theorem 4.2	0.0028	293	0.016	0
Theorem 4.2	0.0028	293	0.015	0.0798

Table 2
Example 5.2: stabilizing T and p for digital implementation.

Theorems	T	p	η_M	σ_1
Theorem 3.2	0.0098	32	0.00000001	0
Theorem 4.2	0.0097	32	0.0000259	0
Theorem 4.2	0.0097	32	0.0000250	0.0028

Simulations of the solutions of the deterministic sampled-data system confirm the theoretical results. Moreover, choosing $\eta_M = 0.00027$ and $p = 43$ we find that the system preserves asymptotic stability for larger than the theoretically predicted $T = 0.0195$ (till $T \approx 0.027$). Note also that for large $T = 0.027$ and $\eta_M = 0.00027$ the system preserves stability for all $p = 5, 6, \dots, 43$.

Remark 5.1. In Karafyllis (2008), by using another design for the chain of three integrators, a theoretical bound on h of the order of 10^{-3} was obtained leading to high controller gains. Simulations in Karafyllis (2008) showed that convergence was preserved for much larger delays with $h \leq 0.21$ illustrating essential conservatism of the theoretical results.

Example 5.2. Consider

$$A_0 = A_2 = 0, \quad A_1 = -1, \tag{5.7}$$

and (5.5). Here Theorems 3.1 and 4.1 give stability intervals $h \in [0.005, 0.585]$ and $h \in [0.015, 0.544]$ respectively, i.e. Theorem 3.1 is less conservative. For sampled-data case, the results that follow from Theorems 3.2 and 4.2 are given in Table 2. Here Theorems 3.2 and 4.2 lead to close results. Simulations of the solutions of the deterministic sampled-data system show that for $\eta_k \equiv \eta_M = 0.000025$ and $p = 32$ the system is asymptotically

stable for $T = 0.0097$ (as theoretically predicted), and it preserves stability for larger T (till $T \approx 0.03$).

5.2. Scalar system of the fourth order

In the present section we consider (2.1) with $n = 4$, $k = m = 1$ and $B = 1$. For systems of the order $n \geq 4$ it is difficult to find directly K that leads to Hurwitz D . So, here we will first find \bar{K} such that $D = A + B\bar{K}$ is Hurwitz (e.g. by using pole placement), and then find $K = \bar{K}M^{-1}$.

Example 5.3. Consider the chain of four integrators, i.e. (2.1) with $A_i = 0$ ($i = 0, \dots, 3$). Choose

$$\bar{K} = [-0.0208 \quad -0.3200 \quad -1.1800 \quad -0.7000] \quad (5.8)$$

that leads to the eigenvalues $[-0.2 + i, -0.2 - i, -0.1, -0.2]$ of $D = A + B\bar{K}$. Let $h_i = ih$ ($i = 0, 1, 2, 3$). The gain K is found as $K = \bar{K}M^{-1}$. Here LMIs of Theorems 3.1 and 4.1 are feasible for the same values of h : $h \in [0.000001, 0.0873]$. Thus, for $h = 0.0873$ we obtain

$$K = [-1368.5 \quad 3941.4 \quad -3781.1 \quad 1208.1].$$

Substituting the latter gain in LMIs of Theorems 3.1 and 4.1, where $h_i = ih$ ($i = 1, 2, 3$) and $D = A + B\bar{K}M$, we find almost the same asymptotic stability intervals $h \in [0.0871, 0.0952]$ and $h \in [0.0873, 0.0952]$ respectively. Simulations of the solutions show that the system is asymptotically stable for a larger interval $h \in [0.074, 0.17]$. So, LMIs are efficient also in this example.

In the presence of noise $A_0 = 0.05\dot{w}(t)$, Theorem 4.2 guarantees asymptotic mean square stability for the interval $h \in [0.0876, 0.0950]$.

6. Conclusion

Static output-feedback controllers with stabilizing artificial delays are attractive due to their simplicity in implementation. However, simple and efficient conditions for the design and robustness analysis of such controllers were missing. The present paper fills this gap introducing simple LMIs for robust stability analysis of the closed-loop systems with multiple delays and justifying that these LMIs are always feasible for small enough delays. Further improvements and extensions to network-based stabilization by using artificial delays may be topics for future research. Some preliminary results on network-based stabilization of systems with relative degree two by using artificial delays are presented in Selivanov and Fridman (2017).

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